Fjrs.futa.edu.ng
FUTA Journal of Research in Sciences
ISSN: 2315 – 8239 (Print); E-ISSN: 2489 - 0413

FUTA Journal of Research in Sciences, Vol. 15(1), April, 2019: 32-39

HYBRID BLOCK ALGORITHM FOR SOLVING DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH HESSENBERG INDEX 3

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ABSTRACT
Higher-Index Differential-Algebraic Equations (DAEs) are known to be numerically and analytically difficult to solve. In this paper, an hybrid block integrator of order seven is proposed for the solution of Hessenberg DAEs of Index-3. This is achieved by constructing a continuous hybrid second derivative method used to obtain the main and additional methods which are combined to form a single block method that simultaneously provide the approximate solutions to the DAEs. The stability analysis of the derived block integrator is discussed. Two test problems are solved to demonstrate the efficiency of the method.

Keywords: Differential-Algebraic Equations, Stability analysis, Hybrid Block Algorithm, Hessengberg DAEs

INTRODUCTION

Differential-Algebraic Equations (DAEs) arise in a variety of applications in various branches of science and engineering which include mathematical modeling, simulation of mechanical systems, optimal control problems, chemical processes, electric circuit design and dynamic systems. Thus, DAEs have attracted the attention of numerical analysts (see Asher, 1989; Brenan *et al.*, 1989; Bayram and Celik, 2004; Benhammonda, 2015). Consider the non-autonomous DAE of the form

$$F(t, y(t), y'(t)) = 0 \qquad (1)$$

on an interval I. If $\frac{\partial F}{\partial y'}$ is non-singular then it is possible to formally solve (1) for y' but if $\frac{\partial F}{\partial y'}$ is singular, the solution y has to satisfy certain algebraic constraints. Thus equation (1) where $\frac{\partial F}{\partial y'}$ is singular is referred to as a Differential Algebraic equation DAE. Celik et al (2002), Celik and Bayram (2003), Celik (2004) and some other aforementioned scholars discussed DAEs of different differential indexes to a large extent. The differential index of a DAE system can be defined along the solution $y(t)$ as the minimal number of differentiations of the DAEs that is required to reduce the DAE system to a set of Ordinary Differential Equations for the variable (Hairer *et al.*, 1992; Karimi and Aminatei, 2011).

In this paper, we consider a special case of (1) which is the semi-explicit DAE of index 3 system of the form

$$x' = f(t, x, y, z), \quad x(t_0) = x_0 \qquad (2a)$$

$$y' = g(t, x, y), \quad y(t_0) = y_0 \qquad (2b)$$

$$0 = h(t, y), \quad z(t_0) = z_0 \qquad (2c)$$

where the product of the three matrix functions $\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\frac{\partial h}{\partial y}$ is non-singular. Most of the higher-index problems encountered in practice can be expressed as a combination of more restrictive structures of ODEs coupled with algebraic constraints. In such system the algebraic and differential variables are identified and all the algebraic variables may be eliminated using the same number of differentiations. The principle of Index reduction is employed to reduce the DAEs to systems of ODEs with initial conditions.

Block methods, first introduced by Milne (1953), have the advantage of being a self-starting method and also have good stability properties. The method simultaneously produces approximation to the solution of Initial Value Problems at a block of points. In addition to having the properties of block method, Hybrid block methods also have the property of utilizing data at points other than the step points $\{t_n | t_n = t_0 + nh\}$. In literature, some numerical methods have been developed for the solution of DAEs such as Backward Differentiation Formula (see Asher, 1989; Brenan *et al.*, 1989; Gear and Petzold, 1984), Implicit Runge-Kutta methods (Brenan *et al.*, 1989), Multibody Solution Algorithms (Pappalardo and Guida, 2018), Newton-Broyden Techniques (Dhamacharoen, 2016), Pade and Chebysev approximation methods (Celik and Bayram, 2003), Pade approximation methods (Ren and Wang, 2017), Variational iterative method (Karka and Celik, 2012) and Block methods (Akinfenwa and Okunuga, 2014; Naghmeh *et al.*, 2012).

In this paper, an Hybrid Block Second Derivative Method (HBSDM) of order seven is developed for the numerical solution of resulting system of ODEs with initial conditions from Hessenberg DAE of Index 3 of the form (2) to yield accurate and efficient results.

DEVELOPMENT OF THE METHOD

Consider the first order IVPs of the form

$$y' = f(t, y), y(t_0) = y_0, \quad t \in [t_0, t_N] \qquad (3)$$

where we assume that the function f is lipchitz continuous and the problem (3) possesses a unique solution. Let y_n be an approximation of the theoretical solution $y(t)$ at t_n. We seek numerical approximation at the points $t_{n+v} = t_n + vh$ and $t_{n+1} = t_n + h$ where h is the step size, n is the grid index and $v \in (0,1)$. The approximation y_n is provided by a continuous approximation $Y(t)$ which is of the form

$$Y(t) = \sum_{j=0}^{7} l_j t^j \qquad (4)$$

where l_j are unknown coefficients. The proposed class of method is thus constructed by imposing the following conditions

$$y(t_{n+i}) = y_{n+i}, \quad i = 0, \frac{1}{3}, \frac{2}{3} \qquad (5)$$

$$y'(t_{n+i}) = f_{n+i}, \quad i = 0, \frac{1}{3}, \frac{2}{3}, 1 \qquad (5)$$

$$y''(t_{n+i}) = g_{n+i}, \quad i = 1 \qquad (5)$$

Equation (5) leads to a system of eight equations which is solved simultaneously to obtain l_j. The Continuous Hybrid Second Derivative Formula (CHSDF) is constructed by substituting the values of l_j into (4) which is simplified and expressed in the form

$$Y(t) = \sum_{j=0}^{k} \alpha_j(t) y_{n+j} + h \sum_{j=0}^{k} \beta_j(t) f_{n+j} + h^2 \rho_k(t) g_{n+k} \qquad (6)$$

where $\alpha_j(t), j = 0, \frac{1}{3}, \frac{2}{3}$; $\beta_j(t), j = 0, \frac{1}{3}, \frac{2}{3}, 1$

and $\rho_1(t)$ are the continuous coefficients. The CHSDF (6) is then used to generate the main method at t_{n+1} while the second derivative of (6) at $t_{n+\frac{1}{3}}$ and $t_{n+\frac{2}{3}}$ is used to generate the additional methods. The developed main and additional methods given in (7(a, b and c)) are combined and implemented simultaneously as a single block method (HBSDM) for the solution of the Hessenberg DAE of Index 3.

$$h^2 g_{n+\frac{1}{3}} = \frac{972}{97} y_n - \frac{4464}{97} y_{n+\frac{1}{3}} + 36 y_{n+\frac{2}{3}} + \frac{632}{873} hf_n - \frac{507}{97} hf_{n+\frac{1}{3}} - \frac{432}{97} hf_{n+\frac{2}{3}} + \frac{259}{873} hf_{n+1} - \frac{25}{873} h^2 g_{n+1} \qquad (7a)$$

$$h^2 g_{n+\frac{2}{3}} = \frac{1107}{194} y_n + \frac{4248}{97} y_{n+\frac{1}{3}} - \frac{99}{2} y_{n+\frac{2}{3}} + \frac{403}{873} hf_n + \frac{750}{97} hf_{n+\frac{1}{3}} + \frac{918}{97} hf_{n+\frac{2}{3}} + \frac{650}{873} hf_{n+1} - \frac{56}{873} h^2 g_{n+1} \qquad (7b)$$

$$y_{n+1}=\frac{16}{97}y_n+\frac{81}{97}y_{n+\frac{1}{3}}+\frac{4}{291}hf_n+\frac{18}{97}hf_{n+\frac{1}{3}}+\frac{36}{97}hf_{n+\frac{2}{3}}+\frac{44}{291}hf_{n+1}-\frac{2}{291}h^2g_{n+1} \qquad (7c)$$

In order to conveniently analyze and implement the derived method (7), we will express it in the block form given in (8)

$$A^{(1)}Y_w=A^{(0)}Y_{w-1}+hB^{(1)}F_w+hB^{(0)}F_{w-1}+h^2C^{(1)}G_w \qquad (8)$$

where $Y_w=\left(y_{n+\frac{1}{3}},y_{n+\frac{2}{3}},y_{n+1}\right)^T$

$$Y_{w-1}=\left(y_{n-\frac{2}{3}},y_{n-\frac{1}{3}},y_n\right)^T$$

$$F_w=\left(f_{n+\frac{1}{3}},f_{n+\frac{2}{3}},f_{n+1}\right)^T$$

$$F_{w-1}=\left(f_{n-\frac{2}{3}},f_{n-\frac{1}{3}},f_n\right)^T$$

$$G_w=\left(g_{n+\frac{1}{3}},g_{n+\frac{2}{3}},g_{n+1}\right)^T$$

Matrices $A^{(0)},A^{(1)},B^{(0)},B^{(1)},C^{(1)}$ are 3 by 3 matrices whose entries are given by the coefficients of equation (8). The matrices $A^{(0)},A^{(1)},B^{(0)},B^{(1)},C^{(1)}$ are defined as follows

$$A^{(1)}=\begin{pmatrix}\frac{4464}{97} & -36 & 0\\ -\frac{4248}{97} & \frac{99}{2} & 0\\ -\frac{81}{97} & 0 & 1\end{pmatrix},$$

$$A^{(0)}=\begin{pmatrix}0 & 0 & \frac{972}{97}\\ 0 & 0 & \frac{1107}{194}\\ 0 & 0 & \frac{16}{97}\end{pmatrix},$$

$$B^{(1)}=\begin{pmatrix}-\frac{507}{97} & -\frac{432}{97} & \frac{259}{873}\\ \frac{750}{97} & \frac{918}{97} & \frac{650}{873}\\ \frac{18}{97} & \frac{36}{97} & \frac{44}{291}\end{pmatrix},$$

$$B^{(0)}=\begin{pmatrix}0 & 0 & \frac{632}{873}\\ 0 & 0 & \frac{403}{873}\\ 0 & 0 & \frac{4}{291}\end{pmatrix},$$

$$C^{(1)}=\begin{pmatrix}-1 & 0 & -\frac{25}{873}\\ 0 & -1 & -\frac{56}{873}\\ 0 & 0 & -\frac{2}{291}\end{pmatrix}$$

ANALYSIS OF HBSDM

In this section, we will discuss the local truncation error, order, zero-stability, consistency, and convergence of the HBSDM.

Local Truncation Error

Following Fatunla (1991) and Lambert (1973; 1991), we define the local truncation error associated with (8) to be the linear difference operator

$$L[y(t);h]=\sum_{j=0}^{k}\alpha_j y(t+jh)-h\sum_{j=0}^{k}\beta_j y'(t+jh)-h^2\gamma_k y''(t+kh) \qquad (9)$$

Assuming that $y(t)$ is sufficiently differentiable, we can expand the terms in (9) as a Taylor series about the point t to obtain the expression

$$L[y(t);h]=C_0y(t)+C_1hy'(t)+C_2hy''(t)+\cdots+C_ph^py^{(p)}(t)+\cdots$$

where the constant coefficients $C_p, p=0,1,\ldots$ are given as follows:

$$C_0=\sum_{j=0}^{k}\alpha_j$$

$$C_1=\sum_{j=0}^{k}(j\alpha_j-\beta_j)$$

$$C_2 = \frac{1}{2!}\left[\sum_{j=0}^{k}(j^2\alpha_j - j\beta_j) - \gamma_k\right]$$

$$\vdots$$

$$C_p = \frac{1}{p!}\left[\sum_{j=0}^{k}(j^2\alpha_j - j\beta_j) - \gamma_k\right]$$

According to Lambert (1973), the method is of order p if $C_0 = C_1 = \cdots = C_p = 0, C_{p+1} \neq 0$. Therefore C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{(p+1)}(t_n)$ is the principal local truncation error at the point t_n. Thus the local truncation error of the method of order p is written as

$$LTE = C_{p+1}h^{p+1}y^{(p+1)}(t_n) + O(h^{p+2})$$

It is established from our calculation that the one-step Block Hybrid Method given in (8) have order p and error constant C_8 given by the vector $p = (7,7,7)^T$ and $C_8 = \left(\frac{61}{178196760}, \frac{17}{39599280}, \frac{1}{59398920}\right)^T$

where T is transpose.

Zero Stability

The zero stability of the HBSDM (8) is determined as the limit h tends to zero and as $h \to 0$, the method tends to the difference system

$$A^{(1)}Y_{w+1} = A^{(0)}Y_w$$

which is normalized to obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \text{Determinant of } (RA^{(1)} - A^{(0)}) = -\frac{68040}{97}R^2(R-1)$$

The block method (10) is zero-stable for $\rho(R) = 0$ and satisfies $|R_j| \leq 1, j = 1, \ldots, k$ and those roots with $|R_j| = 1$, the multiplicity does not exceed 1.

Consistency and Convergence

The block method (8) is consistent since each of the hybrid methods has order $p > 1$. According to

Henrici (1962), convergence = consistency + zero-stability. Hence the HBSDM is convergent.

Linear Stability

Applying the HBSDM (8) to the test equations

$$y' = \lambda y, \quad y'' = \lambda^2 y, \quad \lambda > 0$$

yields

$$Y_w = Q(z)Y_{w-1}, \quad z = \lambda h$$

where the amplification matrix $Q(z)$ is given by

$$Q(z) = \left(A^{(1)} - zB^{(1)} - z^2C^{(1)}\right)^{-1}\left(A^{(0)} + zB^{(0)}\right)$$

The matrix $Q(z)$ has eigenvalues $\{\tau_1, \tau_2, \tau_3\} = \{0, 0, \tau_3\}$ where the dominant eigenvalue τ_3 is the stability function which is a rational function with real coefficient given by

$$R(z) = \frac{701.443 + 491.043z + 68.05z^2 + 6.09023z^3 + 0.41237z^4 + 0.0137457z^5}{701.443 - 405.641z + 97.7503z^2 - 17.2587z^3 + 2.01934z^4 - 0.151203z^5 + 0.00687285z^6}$$

The stability domain of the method S is defined as $S = \{z \in C : R(z) \leq 1\}$. The proposed method HBSDM is L_0-stable since the stability region do not cover the entire left plane of the graph (A_0-stability) and the limit of the stability function $R(z)$ is zero as $z \to \infty$. The stability region for the HBSDM is given in Figure 1 showing the zeros and the poles of the stability function $R(z)$.

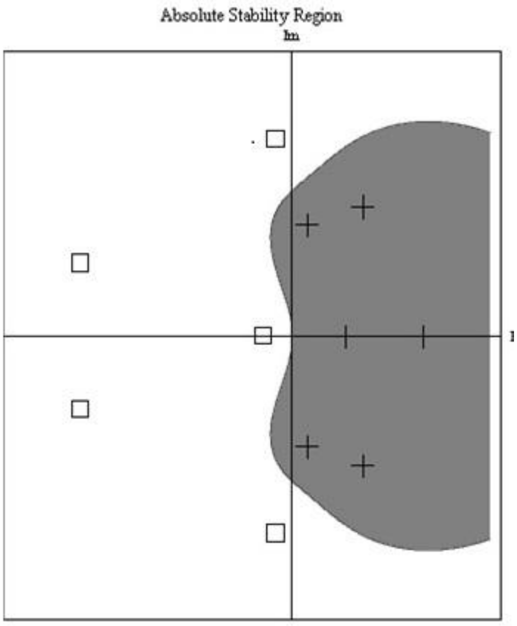

Figure 1: Region of Absolute Stability (RAS) of the HBSDM

NUMERICAL RESULTS

In this section, two experiments are presented to demonstrate the capacity of the proposed method. The computations associated with the experiment discussed were carried out in Mathematica 9.0 on a PC.

Experiment 4.1 Consider the following DAE of Index-3 problem also solved by Karka and Celik (2012) and Akinfenwa (2014)

$$y_1' + y_1 + y_2 + ty_3 - 2t = 0$$

$$y_2' + e^t y_1 + (t+1)y_2 - t^2 - t - 2 = 0$$

$$t^2 y_2 - t^3 = 0$$

with initial conditions

$$y_1(0) = y_3(0) = 1, \quad y_2(0) = 0$$

whose exact solution is given as

$$y_1(t) = e^{-t}, \quad y_2(t) = t, \quad y_3(t) = 1$$

where y_1 and y_2 represent the differential variables and y_3 is the algebraic variable. The DAEs system above is of Index 3 since three times of differentiation of the algebraic part of the DAE will lead to a system of ODEs and they are

$$y_1' = -y_1 - y_2 - ty_3 + 2t$$

$$y_2' = -e^t y_1 - (t+1)y_2 + t^2 + t + 2$$

$$y_3' = 0$$

The numerical results of experiment 1 are tabulated for $h = 0.1$ in Table 1 and compared with the results of Akinfenwa (2014) of order 7 and Karka and Celik (2012).

Experiment 4.2 Consider the following DAE of Index-3 problem also solved by Karka and Celik (2012) and Akinfenwa (2014)

$$y_2' + y_1 - 1 = 0$$

$$ty_2' + y_3' + 2y_2 - 2t = 0$$

$$ty_2 = y_3 - e^t$$

with initial conditions

$$y_1(0) = 0, \quad y_2(0) = -1, \quad y_3(0) = 1$$

whose exact solution is given as

$$y_1(t) = e^t - 1, \quad y_2(t) = 2t - e^t, \quad y_3(t) = (1+t)e^t - 2t^2$$

After three times of differentiation of the given DAE, we have the following system of ODEs

$$y_1' = e^t$$

$$y_2' = 2 - e^t$$

$$y_3' = ty_1 - 2y_2 + t$$

Which confirms that the DAE is of Index 3. Tables 2 – 4 shows the comparison of different methods for $y_1^*(t)$, $y_2^*(t)$ and $y_3^*(t)$ respectively.

Table 1: Comparison of methods for $y_1^*(t)$ in experiment 4.1

t	Error in Karka (2012)	Error in BSDF (2014)	Error in HBSDM
0	0	0	0
0.1	1.00×10^{-10}	6.70×10^{-11}	3.33×10^{-16}
0.2	1.00×10^{-10}	4.50×10^{-11}	8.88×10^{-16}
0.3	1.00×10^{-10}	5.10×10^{-11}	7.77×10^{-16}
0.4	1.00×10^{-10}	4.03×10^{-11}	9.99×10^{-16}
0.5	2.00×10^{-10}	4.50×10^{-11}	1.11×10^{-15}
0.6	1.60×10^{-9}	8.30×10^{-11}	1.44×10^{-15}
0.7	7.20×10^{-9}	6.92×10^{-11}	1.67×10^{-15}
0.8	2.76×10^{-8}	7.17×10^{-11}	1.94×10^{-15}
0.9	8.87×10^{-8}	6.60×10^{-11}	1.61×10^{-15}
1.0	2.52×10^{-7}	6.88×10^{-11}	1.72×10^{-15}

Exact solution of $y_2(t)$ = Numerical solution $y_2^*(t) = t$

Exact solution of $y_3(t)$ = Numerical solution $y_3^*(t) = 1$

Table 2: Comparison of methods for $y_1^*(t)$ in experiment 4.2

t	Error in Karka (2012)	Error in BSDF (2014)	Error in HBSDM
0	0	0	0
0.1	1.00×10^{-10}	1.20×10^{-10}	7.77×10^{-16}
0.2	2.00×10^{-10}	9.10×10^{-11}	1.55×10^{-15}
0.3	4.00×10^{-10}	1.06×10^{-10}	2.66×10^{-15}
0.4	3.00×10^{-10}	9.40×10^{-11}	3.66×10^{-15}
0.5	5.00×10^{-10}	1.05×10^{-10}	4.44×10^{-15}
0.6	1.30×10^{-9}	3.04×10^{-10}	5.33×10^{-15}
0.7	8.00×10^{-9}	2.55×10^{-10}	6.66×10^{-15}
0.8	3.10×10^{-8}	2.80×10^{-10}	7.55×10^{-15}
0.9	1.05×10^{-7}	2.59×10^{-10}	8.88×10^{-15}
1.0	3.01×10^{-7}	2.79×10^{-10}	1.04×10^{-14}

Table 3: Comparison of methods for $y_2^*(t)$ in experiment 4.2

t	Error in Karka (2012)	Error in BSDF (2014)	Error in HBSDM
0	0	0	0
0.1	1.00×10^{-10}	1.20×10^{-10}	6.66×10^{-16}
0.2	2.00×10^{-10}	9.10×10^{-11}	9.99×10^{-16}

0.3	4.00×10^{-10}	1.06×10^{-10}	1.78×10^{-15}
0.4	3.00×10^{-10}	9.40×10^{-11}	2.22×10^{-15}
0.5	5.00×10^{-10}	1.05×10^{-10}	2.66×10^{-15}
0.6	1.30×10^{-9}	3.04×10^{-10}	3.55×10^{-15}
0.7	7.80×10^{-9}	2.55×10^{-10}	4.88×10^{-15}
0.8	3.13×10^{-8}	2.80×10^{-10}	5.77×10^{-15}
0.9	1.04×10^{-7}	2.59×10^{-10}	7.10×10^{-15}
1.0	3.02×10^{-7}	2.79×10^{-10}	8.66×10^{-15}

Table 4: Comparison of methods for $y_3^*(t)$ in experiment 4.2

t	Error in Karka (2012)	Error in BSDF (2014)	Error in HBSDM
0	0	0	0
0.1	0.00	1.14×10^{-10}	5.55×10^{-15}
0.2	1.00×10^{-9}	8.87×10^{-10}	1.13×10^{-14}
0.3	1.00×10^{-9}	1.05×10^{-10}	1.87×10^{-14}
0.4	1.00×10^{-9}	9.54×10^{-10}	2.64×10^{-14}
0.5	3.00×10^{-9}	1.08×10^{-9}	3.51×10^{-14}
0.6	1.90×10^{-8}	3.10×10^{-9}	4.49×10^{-14}
0.7	9.10×10^{-8}	2.69×10^{-9}	5.68×10^{-14}
0.8	3.53×10^{-7}	3.01×10^{-9}	6.97×10^{-14}
0.9	1.16×10^{-6}	2.88×10^{-9}	8.48×10^{-14}
1.0	3.31×10^{-6}	3.15×10^{-9}	1.01×10^{-13}

The numerical results of experiments 4.1 and 4.2 show that the Hybrid Block Second Derivative Method (HBSDM) works excellently for solving DAEs of Index 3. It performed better than the BSDF method of Akinfenwa (2014) and Karka and Celik (2012).

CONCLUSION

In this paper, an Hybrid Block Second Derivative Method (HBSDM) of order seven has been proposed to solve Differential Algebraic Equations with Hessenberg Index 3. The proposed algorithm is based on interpolation and collocation techniques. The method has successfully handled the two examples treated with the help of index reduction principles to convert the DAEs to system of IVPs. Details of the numerical results are displayed in Tables 1-4. The results reveal that the method is appropriate for the numerical solution of this class of problems being considered.

REFERENCES

Akinfenwa O. A. and Okunuga S. A. (2014). Block Method for the Numerical Solution of Differential-Algebraic Equations with Hesenberg index-3, International Journal of Pure and Applied Mathematics, Vol. 93(2), 201-215.

Ascher, U. M. (1989). On Symmetric Schemes and Differential- Algebraic Equations, SIAM Journal on Scientific Computing, Vol. 10(5) 937 - 949.

Bayram, M. and Celik E. (2004). Chebyshev Approximation for numerical solution of Differential-Algebraic Equations, International Journal of Applied Mathematics and Statistics, Vol. 2 (1) 29-36.

Benhammonda B. (2015). Solution of nonlinear higher index Hessenberg DAEs by Adomian Polynomials and Differential Transform Method, SpringerPlus (2015)4:648. https://doi.org/10.1186/540064-015-1443-3.

Brenan K. E., Campbell S. L. and Petzold L. R. (1989). Numerical Solution of Initial Value Problems in Differential-Algebraic Equations, Elsevier, New York.

Celik E. (2004). On the numerical solution of Differential-Algebraic Equations with index- 2, Appl. Math. Comput., 156, 541-550.

Celik E. and Bayram, M. (2003). On the numerical solution of Differential-Algebraic Equations by Pade Series, Applied Mathematics and Computations, 137 (1) 151-160.

Celik E., Karaduman E. and Bayram, M. (2002). Numerical method to solve Chemical Differential-Algebraic Equations, International Journal of Quantum Chemistry, 89, 447-451.

Dhamacharoen A. (2016). Efficient Numerical Methods for solving Differential-Algebraic Equations, Journal of Applied Mathematics and Physics, 4(1), 39-47.

Fatunla S. O. (1991). Block methods for second order Initial Value Problems, International Journal for Comput. Math. Vol. 41, 55-63.

Gear, C. W. and Petzold, L. R. (1984).Ordinary Differential Equation systems for the solution of Differential-Algebraic systems, SIAM Journal on Numerical Analysis, Vol.21 (4) 716-728.

Hairer, E., Norsett, S. P. & Wanner G. (1992). Solving Ordinary Differential Equation systems II. Stiff and Differential-Algebraic systems}, Springer, New York.

Henrici P. (1962) Discrete Variable Methods in Ordinary Differential Equations, John Wiley, New York.

Karimi -Vanani, S. and Aminataei, A. (2011). Numerical solution of Differential-Algebraic Equations using a Multiquadric Apporoximation Scheme, Mathematical and Computing Modelling, Vol. 53, 659-666.

Karka M. and Celik E. (2012). On the numerical solution of Differential-Algebraic Equations with Hesenberg index-3, Discrete Dynamics in Nature and Society, Vol 2012 (2012), Article ID 147240, 12 pages. http://dx.doi.org/$10.1155/2012/147240.

Lambert J. D. (1973). Computational methods in Ordinary Differential Equations}, John Wiley and sons, New York.

Lambert J. D. (1991). Numerical Methods for Ordinary Differential Systems}, John Wiley and sons, New York.

Milne W. E. (1953). Numerical solution of Differential Equations}, John Wiley and sons.

Naghmeh A., Mohamed S., Zarina B. I. And Fudziah I. (2012). 2-point and 3-point BBDF Methods for solving Semi-Explicit index-1 DAEs, Applied Mathematical Sciences. Vol. 6 (134) 6679-6689.

Pappalardo C. M. and Guida D. (2018). On the computational Methods for solving the Differential-Algebraic Equations of motion of Multibody Systems, Machines 2018, 6(2), 20.

Ren L. and Wang Y. (2017). Numerical Methods for a class of Differential-Algebraic Equations, Mathematical Problems in Engineering, Hindawi 2017(1), 1-10.

https://doi.org/10.1155/2017/1871590.