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BLOCK MULTISTEP METHOD FOR THE DIRECT SOLUTION OF THIRD ORDER OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper presented the direct block multistep method for solving third order initial value problems in ordinary differential equations. Method of collocation and interpolation of power series approximate solution was used to derive a continuous linear multistep method. Block method was later used to generate the non-overlapping solution at selected grid points. The method is self-starting, not requiring developing separate predictors to implement it and it is better than the conventional predictor-corrector (P-C) methods. Of great interest are some basic properties of the new block multistep method, such as, convergence, order, error constant and zero-stability. These basic properties were investigated. The performance of the new block method was tested with some third order initial value problems.

Keywords: Linear multistep methods (LMMs); Zero-stability; Third order; Initial Value Problems (IVPs); Ordinary Differential Equations (ODEs); Interval of periodicity; Predictor-corrector (P-C)

INTRODUCTION

Consider the nth order initial value problems in ordinary differential equation of the form:

$$y^{(n)} = f(x,\ y,\ y',\ y'' \dots\dots\dots\dots\dots y^{(n-1)}) \quad , \quad (1)$$

$$y(a) = y_0, \dots, y^i(a) = y_i, i = 1(1)n-1, n \geq 3 \quad .$$

assuming that the numerical solution is required on a given set of mesh,

$$\prod = \{x_n / x_n = a + nh, h = x_{n+1} - x_n, n = 0,1,\dots\dots N\}$$

where N = (b-a) / h.

This class of problems (with the absence of derivatives on right hand side) has a lot of applications in the fields of science and engineering and some other areas. The reduction of (1) to system of first order equations will lead to a greater computational cost, hence, we resort to numerical methods. The purpose of this present paper was to develop an alternative approach based on the block linear multistep method for the direct solution of third order ordinary differential equations.

There are considerable literature on the methods of solution to higher order Ordinary Differential Equations (ODEs) by predictor-corrector methods (Lambert, 1973; 1991; Onumanyi *et al.*, 1994; Fatunla, 1994 ; Awoyemi, 2003; Awoyemi and Idowu, 2005; Adee *et al.*, 2005). These methods have certain limitations; the computer programmes associated with the methods are often complicated especially when incorporating subroutines to supply the starting values for the methods, thus resulting in longer computer time and more computational work (Jator, 2007).

Recently, authors adopted block methods for solving higher order ODEs (Jator, 2007; Olabode and Yusuph, 2009; Siamak, 2010; Awoyemi *et al.*, 2011). In particular, Olabode and Yusuph (2009) developed a new block method for special third order ODEs for step number k equals three and which was better in accuracy and efficiency than Awoyemi (2003). In Jator (2007), a class of initial value methods for the direct solution of second order initial value problems were constructed, linear multistep methods with continuous coefficients were obtained and applied as simultaneous numerical integrators to $y'' = f(x, y, y')$. The implementation strategy is more efficient than those given in Awoyemi (1999) which are applied over overlapping intervals in predictor-corrector mode.

Moreover, Yap *et al.*, (2008) developed block methods based on Newton interpolation for solving special second other ODEs directly. Majid *et al.*, (2010) derived variable step size block method for solving directly third order ODEs. Majid *et al.*, (2012) constructed two-point four

step block method for the solution of general third order ODEs.
Therefore, this paper proposes the block multistep method of step number k equals six, for the direct solution of third order initial value problems of ordinary differential equations.

MATERIAL AND METHODS

In this section, before describing the method, the theorems that establish the existence and uniqueness of the solutions of higher order ordinary differential equations are stated.

Theorem 2.1: (Wend, 1967).
Given the general nth order initial value problem (1).
Let R be the region defined by the inequalities
$0 \le x - x_0 \le a, |r_k - y_k| < b_k, k = 0, 1, \ldots, n-1$, where $y_k \ge 0$ for $k > 0$.
Suppose the function $f(x, r_0, r_1, r_2 \ldots, r_{n-1})$ in equation (1) is non-negative, continuous and non-decreasing in x and r_k, $k = 0, 1, 2 \ldots, n-1$ in the region R.
If in addition $f(x, y_0, y_1, y_2, \ldots, y_{n-1}) \ne 0$ in R for $x > x_0$ then, equation (1) has at most one solution in R.
Theorem 2.2: (Wend, 1967).
Let $w_{(n)} = f(x, w, w', \ldots, w^{(n-1)}), w^{(k)}(x_0) = c_k,$ (2a)
where $k = 0, 1, 2, \ldots, (n-1)$, w and f are scalars. Let R be the region defined by the inequalities $x_0 \le x_0 + a, |r_j - c_j|$, $j - 0, 1, \ldots n-1, (a > 0, b > 0)$. Suppose the function $f(x, r_0, r_1, r_2 \ldots, r_{n-1})$ is defined in R and in addition:

(a) f is non-negative and non-decreasing in each of $x, r_0, r_1, r_2 \ldots, r_{n-1}$ in R;
(b) $f(x, c_0, c_1, c_2, \ldots, c_{n-1}) > 0$ for $x_0 \le x \le x_0 + a$ and
(c) $c_k \ge 0, k = 0, 1, 2, \ldots. n-1$.

Then, (2a) has a unique solution in R.
For the purpose of this research work, we shall consider the ODE of the type

$$y'''(x) = f(x, y), \quad y(a) = y_0, \quad y'(a) = \eta_0, \quad y''(a) = \eta_1 \tag{2b}$$

Moreover, a power series of a single variable x in the form:

$$P(x) = \sum_{j=0}^{\infty} a_j x^j \tag{3}$$

is used as the basis or trial function, to produce the approximate solution as

$$y(x) = \sum_{j=0}^{k+1} a_j x^j \tag{4}$$

$$a_j \in R, j = 0(1)k+2, y \in C^m(a,b) \subset P(x).$$

Assuming an approximate solution to (1) in the form of (4) whose high derivatives

$$y'(x) = \sum_{j=0}^{k+1} j(j-1) a_j x^{j-1} \tag{5}$$

$$y''(x) = \sum_{j=0}^{k+1} j(j-1)(j-2) a_j x^{j-2} \tag{6}$$

$$y'''(x) = \sum_{j=0}^{k+1} j(j-1)(j-2)(j-3) a_j x^{j-3} \tag{7}$$

.....

.....

....

$$y^n(x) = \sum_{j=0}^{k+1} j(j-1)(j-2)(j-3)\ldots.(j-(n-1))(j-n) a_j x^{j-n} \tag{8}$$

From (1) and (8) one obtains:

$$f(x,y,y',y''.......y^{n-1}) == \sum_{j=0}^{k+1} j(j-1)(j-2)(j-3)....(j-(n-1))(j-n)a_j x^{j-n} \qquad (9)$$

where $a_j's$ are the parameters to be determined. For step number six, equation (9) was collocated at the grid-points $x = x_{n+j}, j = 0(1)k$ and equation (4) was interpolated at 0, 1, 2.

$$\mathrm{f}_{n+j} = \sum_{j=0}^{k+1} j(j-1)(j-2)(j-3)....(j-(n-1))a_j {x_{n+j}}^{j-n} \qquad (10)$$

$$\mathrm{y}_{n+j} = \sum_{j=0}^{k+1} a_j {x_{n+j}}^{j-n}, j = 0,\ 1, 2 \qquad (11)$$

In Jator (2007), matrix inversion approach was employed in the determination of the unknown parameters. Putting (10) and (11) in the form of matrix equation and then solving the resulting equations so as to obtain parameters a_j, yields, after some manipulation, the new continuous method

$$y(x) = \sum_{j=0}^{2} a_j(x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (12)$$

It was then applied as simultaneous numerical integrator in sequential mode to solve third order ordinary differential equations. The method eliminates the use of predictors by providing sufficiently accurate simultaneous difference equations from a single continuous formula and its derivative. Moreover, this method is cheaper to implement since it is self-starting and therefore the limitations are circumvented.

The coefficients $\alpha_j(x)$ *and* $\beta_j(x)$ of (12) are expressed as functions of t

$$t = \frac{x - x_{n+1}}{h}$$

and these are:

$$\alpha_0 = \frac{1}{2}(t^2 - t)$$

$$\alpha_1 = (-t^2 + 1)$$

$$\alpha_2 = \frac{1}{2}(t^2 + t)$$

$$\beta_0 = \frac{h^3}{3628800}\{10t^8 - 225t^8 + 2040t^7 - 9450t^6 + 23016t^5 - 25200t^4 + 34875t^2 - 25066t\}$$

$$\beta_1 = \frac{h^3}{3628800}\{-60t^9 + 1260t^8 - 10080t^7 + 35280t^6 - 24696t^5 - 194040t^4 + 604800t^3 + 157500t^2 - 569964t\}$$

$$\beta_2 \frac{h^3}{3628800}\{150t^9 - 2925t^8 + 20520t^7 - 52290t^6 - 42840t^5 + 378000t^4 - 322785t^2 + 22170t\}$$

$$\beta_3 = \frac{h^3}{3628800}\{-200t^9 + 3600t^8 - 22080t^7 + 40320t^6 + 78960t^5 - 252000t^4 + 208080t^2 - 56680t)$$

$$\beta_4 = \frac{h^3}{3628800}\{150t^9 - 2475t^8 + 13320t^7 - 18270t^6 - 47880t^5 + 126000t^4 - 105255t^2 + 34410\}$$

$$\beta_5 = \frac{h^3}{3628800}\{-60t^9 + 900t^8 - 4320t^7 + 5040t^6 + 15624t^5 - 37800t^4 + 31860t^2 - 11244t\}$$

$$\beta_6 = \frac{h^3}{3628800}\{10t^9 - 135t^8 + 600t^7 - 630t^6 - 2184t^5 + 5040t^4 - 4275t^2 + 1574t\} \qquad ....(13)$$

The discrete scheme form of (13) is:

$$y_{n+6} - 15y_{n+2} + 24y_{n+1} - 10y_n = \frac{h^3}{30240}(2341f_n + 141564f_{n+1} + 257505f_{n+2} + 123280f_{n+3} + 66615f_{n+4} + 13044f_{n+5} + 451f_{n+6}) \quad (14)$$

with order $p = 8$ and the error constant $C_{10} = -3.373015873\lambda^{-03}$

where $f_k = f(x_k, y_k)$, $k = 0(1)n + 6$

The first and second derivatives of (14) were also found.

SPECIAL APPLICATIONS OF THE NEW BLOCK LMM

Here, special application of the method is discussed. The evaluation of (13) at $x = x_{n+3}$, $x = x_{n+4}$ and $x = x_{n+5}$, $x = x_{n+6}$ yield, respectively, the four integrators:

$$y_{n+6} - 15y_{n+2} + 24y_{n+1} - 10y_n = \frac{h^3}{30240}(2341f_n + 141564f_{n+1} + 257505f_{n+2} + 123280f_{n+3} + 66615f_{n+4} + 13044f_{n+5} + 451f_{n+6})$$

with order $p = 8$ and error constant $C_{10} = -3.37301587e^{-03}$

$$y_{n+5} - 10y_{n+2} + 15y_{n+1} - 6y_n = \frac{h^3}{30240}(1391f_n + 84894f_{n+1} + 145605f_{n+2} + 52820f_{n+3} + 18345f_{n+4} - 786f_{n+5} + 131f_{n+6})$$

with order $p = 8$ and error constant $C_{10} = -2.248677249e^{-03}$

$$y_{n+4} - 6y_{n+2} + 8y_{n+1} - 3y_n = \frac{h^3}{30240}(695f_n + 42324f_{n+1} + 65739f_{n+2} + 10544f_{n+3} + 2109f_{n+4} - 516f_{n+5} + 65f_{n+6})$$

with order $p = 8$ and error constant $C_{10} = -1.124338624e^{-03}$

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{30240}(221f_n + 14109f_{n+1} + 16986f_{n+2} - 1774f_{n+3} + 921f_{n+4} - 255f_{n+5} + 32f_{n+6}) \quad (15)$$

$$hZ_{n+6} + \frac{1}{2}y_{n+2} - 2y_{n+1} + \frac{3}{2}y_n = \frac{h^3}{1814400}(96967f_n + 644628f_{n+1} - 288165f_{n+2} + 251440f_{n+3} - 137235f_{n+4} + 43068f_{n+5} - 5903f_{n+6})$$

with order $p = 8$ and error constant $C_{10} = -2.609126984e^{-03}$.

$$h^2Z'_{n+6} - y_{n+2} + 2y_{n+1} - y_n = \frac{h^3}{1814400}(-537735f_n - 1795860f_{n+1} + 1071045f_{n+2} - 917040f_{n+3} + 501075f_{n+4} - 157500f_{n+5} + 21615f_{n+6})$$

with order $p = 8$ and error constant $C_{10} = 9.628527337e^{-03}$ (16)

The basic properties of the new block multistep method, such as convergence, order, error constant and zero-stability were determined. The method was found to be consistent and convergent with order $p = 8$ and error constant $C_{10} = -3.373015873\lambda^{-03}$. Following Fatunla (1994) and Lambert (1973), the linear difference operator L is defined as:

$$L[y(x);h] = \sum_{j=0}^{k}[\alpha_j y(x + jh) - h^4\beta_j y^{iv}(x + jh)] \quad (17)$$

where y(x) is the exact solution to (1) and is assumed to be sufficiently differentiable. We now invoke the Taylor's theorem to obtain

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + \dots + C_q h^q y^{(q)}(x) + o(h^{q+2}) \quad (18)$$

whose coefficients C_q, q = 0,1,.. are constants independent of y(x) are given as:

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j$$

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)\sum_{j=1}^{k} j^{q-2}\beta_j\right] \quad (19)$$

The order p of the difference operator L[y(x); h] is a unique integer p such that $C_q = 0$, q = 0(1) p+1, $C_{P+2} \neq 0$ (Henrici 1962).

For convenience and in order to determine the zero stability of the method, equations (15) and (16) are written as block method given by the matrix difference equation. The first-block of linear multistep method for the third order initial value problem designated by equation (1) can be expressed by the following matrix difference equation:

$A^{(o)}. y_q = A^{(1)}.y_{q-1} + h^3 B^{(0)}.F_q + BF_{q-1}$ (20)

where, $y_q = (y_{n+1}, y_{n+2,\dots\dots}, y_{n+6})^T, y_{q-1} = (y_{n-5}, y_{n-4,\dots\dots\dots}, y_n)^T$

$f_{q-1} = (f_{n-5}, f_{n-4},\dots\dots, f_n)^T$, $f_q = (f_{n+1}, f_{n+2,\dots\dots}, f_{n+6.})^{T,}$ q = 0, 1, ...

and n= 0, 6….and the matrix A^o is an identity matrix

NUMERICAL EXPERIMENTS AND RESULTS

This section deals with the implementation of some algorithms proposed for problem (1)

1. $y''' = 3\sin x$

$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2$

Theoretical solution is

$$y(x) = 3\cos x + \frac{x^2}{2} - 2$$

Table 1: **Comparison of absolute errors of Block multistep method of order eight and those of Predictor-Corrector method of the same order eight.**

X	Exact solution y(x)	y-computed	Errors in block multistep method of order Eight	Errors in Predictor-corrector method of same order eight
0.1	0.990012496	0.990012496	1.65922E-10	4.172279744E-09
0.2	0.960199734	0.960199734	4.76275E-10	9.578546178E-08
0.3	0.911009467	0.911009468	6.23182E-10	3.991586710E-07
0.4	0.843182982	0.843182984	19.9134E-10	1.036864440E-06
0.5	0.757747686	0.757747686	3.28882E-10	2.128509889E-06
0.6	0.656006845	0.656006846	1.27096E-09	3.789539851E-06
0.7	0.539526562	0.539526567	4.84653E-09	6.130086676E-06
0.8	0.410120128	0.410120139	1.09585E-08	9.253867047E-06
0.9	0.269829905	0.269829925	2.0188E-08	1.325714643E-05
1.0	0.120906918	0.120906953	3.53956E-08	1.822777782E-05
1.1	-0.034211636	-0.034211579	5.66233E-08	2.424432295E-05
1.2	-0.192926737	-0.192926653	8.35700E-08	3.137526880E-05

It could be observed in table 1, that the block multistep method of order eight is more accurate than the predictor-corrector method of order eight for problem 1.

2. $y''' = e^x$

$y(0) = 3, \quad y'(0) = 1, \quad y''(0) = 5$

Theoretical solution is

$y(x) = 2 + 2x^2 + e^x$

Table 2: **Comparison of absolute errors of Block LMM of order eight and those of Predictor-Corrector Method of the same order for Problem 2, h = 0.1.**

X	Exact solution y(x)	Block multistep method y-computed	Errors Block in block multistep method	Errors in Predictor-Corrector Method of order eight
0.1	3.125170918	3.125170919	9.24352E-10	1.403538619E-09
0.2	3.301402758	3.30140276	18.3983E-10	3.269138249E-08
0.3	3.529858808	3.52985881	24.2400E-10	1.395151714E-07
0.4	3.811824698	3.811824703	53.5873E-10	3.723331807E-07
0.5	4.148721271	4.14872127	7.00128E-10	7.869058836E-07
0.6	4.54211885	4.5421188	3.90509E-10	1.444874331E-06
0.7	4.993752707	4.993752714	6.52952E-09	2.414343780E-06
0.8	5.505540928	5.50554095	2.15075E-08	3.770586333E-06
0.9	6.079603111	6.07960315	3.88430E-08	5.596789306E-06
1.0	6.718281828	6.71828189	6.15410E-08	7.984910299E-06
1.1	7.424166024	7.424166114	9.00536E-08	1.103654222E-05
1.2	8.200116923	8.20011705	1.27263E-07	1.486399811E-05

The direct block multistep method of order eight is more accurate than when it was implemented in predictor-corrector mode as shown in table 2 above.

Problem 3

$y''' = -y$

$y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1$

Theoretical solution is

$y(x) = e^{-x}$

Table 3: **Absolute Errors For The Block LMM for Problem 2, h=0.1**

X	Exact solution y(x)	y-computed	Errors in block multistep method
0.1	0.904837418	0.904837416	1.66845548E-09
0.2	0.818730753	0.818730742	1.07247812E-08
0.3	0.740818221	0.740832811	1459.00969E-08
0.4	0.670320046	0.67031999	5.60600832E-08
0.5	0.60653066	0.606530552	1.07616918E-07
0.6	0.548811636	0.548811456	1.80036532E-07
0.7	0.496585304	0.496581992	3.31228689E-06
0.8	0.449328964	0.449675668	3.46704102E-04
0.9	0.40656966	0.408849308	2.27964863E-03
1.0	0.367879441	0.374259661	6.38022004E-03
1.1	0.332871084	0.345512915	1.26418315E-02
1.2	0.301194212	0.32225658	2.10623676E-02

CONCLUSION

In this paper, the block multistep method of order eight has been developed and implemented in sequential mode. It was more accurate than the predictor-corrector method of the same order as shown in tables 1 and 2. Evaluation of the continuous formula along with its derivatives where necessary, leads to k simultaneous discrete linear multistep methods of comparable convergence properties for a simultaneous application to the ordinary differential equations of the form (1) in block form with fixed or variable step-size, thereby eliminating the requirement of additional starting values from other one-step methods.

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